Solving a convex quadratic maximization problem appearing in some distributionally robust problem

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EUROPT

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Context

$$\begin{aligned} \max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \quad & d^\top y + c^\top x + \|x\|_2^2 \\ \text{s.t.} \quad & Ax + Ty = b, \\ & x \geq 0,\ y \geq 0 \end{aligned}$$

→ Appears as the **subproblem in a Benders' decomposition** for a Distributionally Robust Optimization (DRO) problem.

→ Solving it to optimality provides the **tightest possible upper bound**.

→ The problem is **NP-hard**.

→ Not necessary to solve it at optimality at each iteration!

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Deterministic UC Problem

$\mathcal{M}$: set of units

T: time horizon

$\xi \in \mathbb{R}^T$: demand vector

x_i: commitment variables (binary).

y_i: production variables (continuous).

$$
\begin{aligned}
\min_{x_i, y_i} \quad & \sum_{i \in \mathcal{M}} c_i^\top x_i + \sum_{i \in \mathcal{M}} b_i^\top y_i \\
s.t. \quad & x_i \in X_i, && \forall i \in \mathcal{M} \\
& y_i \in Y_i(x_i) && \forall i \in \mathcal{M} \\
& \sum_{i \in \mathcal{M}} y_i = \xi
\end{aligned}
$$

2-stage UC Problem under uncertainty

Uncertainty on the demand.

2-stage assumption:

- 1^{st} stage: commitment variables (binary)
- 2^{nd} stage: production variables (continuous)

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\begin{aligned}
\min_{x_i, y_i} \quad & c^\top x + b^\top y \\
s.t. \quad & x \in X \\
& y_i \in Y_i(x_i) \qquad \forall i \in \mathcal{M} \\
& \sum_{i \in \mathcal{M}} y_i = \boxed{\xi}\,?
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$$\begin{aligned} \min_{x_i} \quad & c^\top x + Q(x,\xi) \\ s.t. \quad & x \in X \end{aligned}$$

$Q(x,\xi)$: recourse value representing the optimal cost of the second stage, considering which units are on or off (first stage) and the demand ξ

$$Q(x,\xi) = \max_{\alpha,\beta,\gamma} \alpha^\top \xi + \beta^\top x + \gamma = \max_{k \in \mathcal{K}} \alpha_k^\top \xi + \beta_k^\top x + \gamma_k$$
$$\text{s.t.}\, (\alpha,\beta,\gamma) \in \Lambda$$

Distributionnally Robust Optimization

$$\min_{x\in X} \quad c^\top x + \max_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\xi\sim\mathbb{Q}}[Q(x,\xi)]$$

Interest

Both Robust Optimization and Stochastic Optimization are special cases of DRO.

Stochastic Optimization:

$$\min_{x\in X} c^\top x + \mathbb{E}_{\xi\sim\mathbb{P}_0}[Q(x,\xi)]$$

$$\mathcal{P} = \{\mathbb{P}_0\}$$

Robust Optimization:

$$\min_{x\in X} c^\top x + \max_{\xi\in\mathcal{U}} Q(x,\xi)$$

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{B}(\mathcal{U})\}$$

$\Longrightarrow$ **How should we select $\mathcal{P}$?**

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Wasserstein distance-based ambiguity sets

$$\min_{x \in X} \quad c^\top x + \max_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\xi \sim \mathbb{Q}}[Q(x, \xi)]$$

Idea: $\mathcal{P}$ has to include distributions "close" to the empirical one.

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{B}(\mathbb{R}^T) \mid W_p(\mathbb{Q}, \mathbb{P}_0) \leq \theta\}$$

- $\mathbb{P}_0 = \frac{1}{N} \sum_{i \in [N]} \delta_{\zeta_i}$: empirical distribution
- W_p: Wasserstein distance of order p defined with the cost function $c_p(\xi, \zeta) = \|\xi - \zeta\|_p$.

$p = 1$:	$p = 2$:
- Reformulation with linear programs - Actually, it is the same problem as with $\mathcal{P} = \{\mathbb{P}_0\}$.	- Reformulation with non convex quadratic programs. - Leads to better solutions !

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DRO Unit Commitment problem

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$$\begin{aligned} \min_x \quad & c^\top x + \max_{\substack{\mathbb{Q}:\ W_2(\mathbb{Q},\mathbb{P}_0) \leq \theta \quad [\lambda] \\ supp(\mathbb{Q}) \subset \mathbb{R}^T}} \mathbb{E}_{\mathbb{Q}}[Q(x, \xi)] \\ s.t. \quad & x \in X \end{aligned}$$

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$$\begin{aligned} \min_{x, \lambda \geq 0} \quad & c^\top x + \lambda \theta^2 + \frac{1}{N} \sum_{i=1}^{N} \sup_{\xi \in \mathbb{R}^T} \left(Q(x, \xi) - \lambda \|\xi - \zeta_i\|_2^2 \right) \\ s.t. \quad & x \in X \end{aligned}$$

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$$\min_{x, \lambda \geq 0} \quad c^\top x + \lambda\theta^2 + \frac{1}{N} \sum_{i=1}^{N} \max_{\xi \in \mathbb{R}^T, k \in [K]} \left(\alpha_k^\top \xi + \beta_k^\top x + \gamma_k - \lambda \|\xi - \zeta_i\|^2 \right)$$

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$$\begin{aligned}
\min_{x, \lambda \geq 0, z \geq 0, w \geq 0} \quad & c^\top x + w\theta^2 + \frac{1}{N} \sum_{i=1}^{N} z_i \\
s.t. \quad & x \in X \\
& w \leq \frac{1}{\lambda} \\
& z_i \geq \left(\alpha_k^\top \zeta_i + \beta_k^\top x + \gamma_k + \frac{\lambda \|\alpha_k\|_2^2}{4} \right) \quad \forall k \in [K]
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$\implies$ **Benders' algorithm**

Benders algorithm DRO

Master Problem:
Subset of constraints

(x^*, λ^*)

$(\alpha^*, \beta^*, \gamma^*)$

Subproblem for each scenario i:

$$\begin{aligned} \max_{\alpha,\beta,\gamma} \quad & \alpha^\top \zeta_i + \beta^\top x^* + \gamma + \frac{\lambda^* \|\alpha\|_2^2}{4} \\ s.t. \quad & (\alpha, \beta, \gamma) \in \Lambda \end{aligned}$$

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$(\alpha^*, \beta^*, \gamma^*)$

Convex maximization problem (NP-Hard)

Comparison risk-neutral/ DRO subproblems

Time spent on instance with 50 units and 1 random demand
$\implies$ dimension of the quadratic term: $T = 24$

Subproblem Risk-Neutral for each scenario i:

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$\downarrow$

LP

$\downarrow$

$\approx 0.002s$

Subproblem DRO for each scenario i:

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$\downarrow$

Convex maximization problem (NP-Hard)

$\downarrow$

$\approx 0.5s$

- If we have 100 iterations and 100 scenarios, we have to solve 10,000 subproblems.
- **Idea:** Use non-optimal dual solutions!

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$\boxed{\text{Optimal value: } v^*, \qquad \text{Optimal solution: } (x^*, y^*)}$ $\leftarrow$ Optimal cut

Linearization: ("Easy" to solve)

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(\mathcal{P}_\ell(\bar{x})) \quad \begin{aligned} \max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \quad & d^\top y + (c + 2\bar{x})^\top x - \|\bar{x}\|_2^2 \\ \text{s.t.} \quad & Ax + Ty = b, \\ & x \geq 0,\ y \geq 0 \end{aligned}
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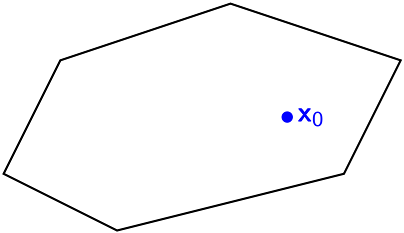
Frank-Wolfe Algorithm

$$f(x, y) = d^\top y + c^\top x + \|x\|_2^2$$



Choice of the step length α_k by solving:

$$\max_{\alpha \in [0,1]} f((x_k, y_k) + \alpha(p_k^x, p_k^y))$$

$\implies$ By convexity, $\alpha_k \in \{0, 1\} \implies (x_{k+1}, y_{k+1}) = s^\ell(x_k)$.

$\implies$ The sequence $(f(x_k, y_k))_{k \geq 1}$ is strictly increasing.

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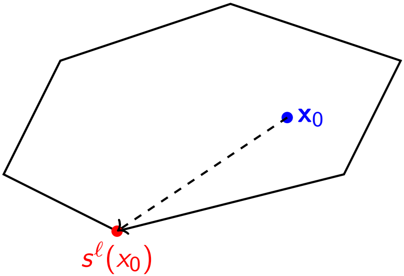

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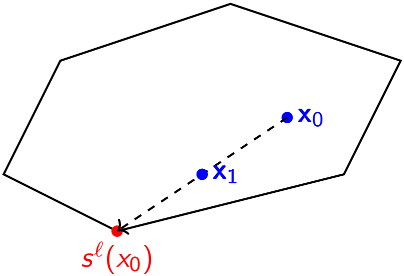

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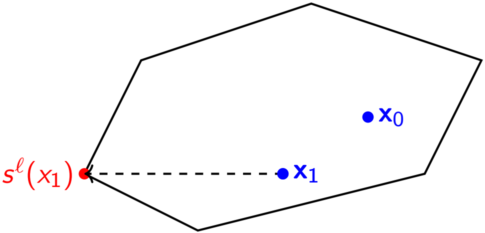

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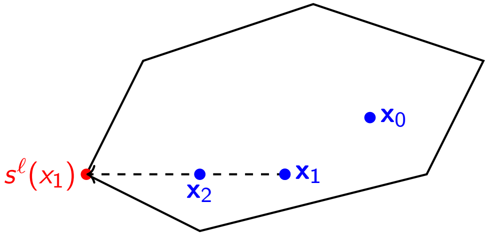

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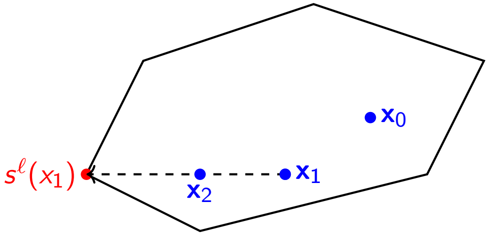

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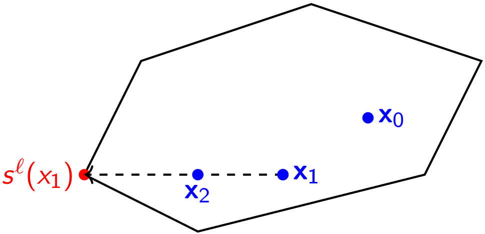

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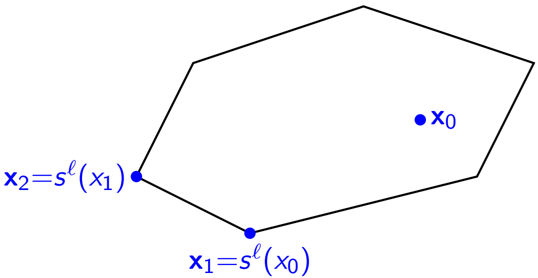

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$$\max_{\alpha \in [0,1]} f((x_k, y_k) + \alpha(p_k^x, p_k^y))$$

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$\implies$ The sequence $(f(x_k, y_k))_{k \geq 1}$ is strictly increasing.

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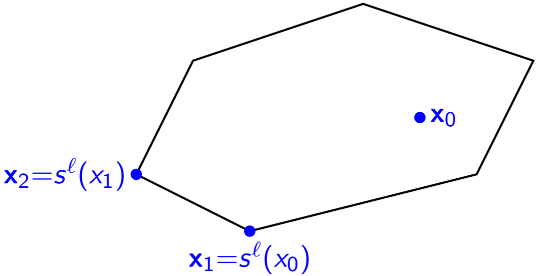
$$f(x,y) = d^\top y + c^\top x + \|x\|_2^2$$



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1^{st} starting point strategy: the non-DRO solution

Linearized problem with $\bar{x} = 0$

$$
\begin{aligned}
\max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \quad & d^\top y + (c + 2\bar{x})^\top x - \|\bar{x}\|_2^2 \\
\text{s.t.} \quad & Ax + Ty = b, \\
& x \geq 0, \; y \geq 0
\end{aligned}
$$

Stochastic Optimization

$$
\begin{aligned}
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$$

- The DRO problem is in general “close” to the empirical one.
- Starting with $x_0 = 0$ is in general a good choice.

2^{nd} starting point strategy: Sampling

Definition (Attraction field of a face F)

The attraction field of a face F at $\bar{x}$ is the set of points $P(F)$ such that the linearized problem at $\bar{x}$ has F as the set of optimal solutions.

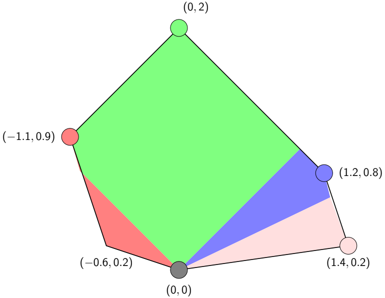

Figure: Maximizing $\|x\|_2^2$

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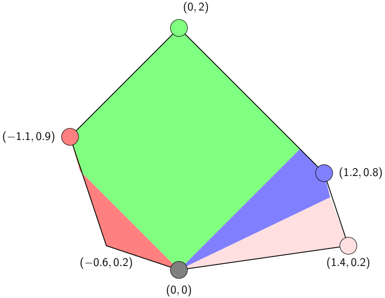
The set $P(F)$ is a **polyhedron**. $P(x^*)$ has a **non-zero measure**. A point is a local optimum if and only if it belongs to its attraction field.



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- FW algorithm finds a local optimum.
- Randomly draw the starting point.

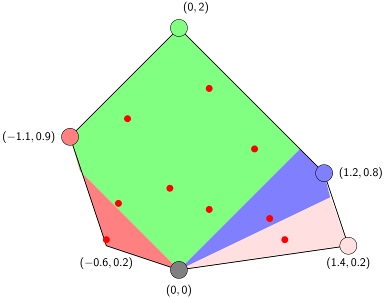

Figure: Maximizing $\|x\|_2^2$

3^{rd} starting point strategy: Discretize the space

Proposition

Let $\bar{x}$ and $v(x,y)$ defined by :

$$v(x,y) = d^\top y + c^\top x + \|x\|_2^2$$

the following inequality holds:

$$v^\ell(\bar{x}) \leq v(s^\ell(\bar{x})) \leq v^* \leq v^\ell(\bar{x}) + \|\bar{x} - x^*\|^2.$$

Corollary

Let $\varepsilon > 0$ and $X \subset \mathbb{R}^n$ such that:

$$\max_{x \in P} dist(x, X) \leq \varepsilon \qquad dist(x, X) = \min_{\bar{x} \in X} \|x - \bar{x}\|_2$$

Define $v^\ell(X) = \max_{x \in X} v^\ell(x)$. Then, the following inequalities hold:

$$v^\ell(X) \leq v^* \leq \varepsilon^2 + v^\ell(X)$$

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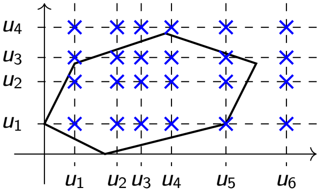

$$\begin{aligned} v^\ell(X) = \max_{\bar{x},x,y} \quad & d^\top y + (c + 2\bar{x})^\top x - \|\bar{x}\|_2^2 \\ \text{s.t.} \quad & Ax + Ty = b, \\ & x \geq 0,\ y \geq 0, \bar{x} \in X \end{aligned}$$

$\Longrightarrow$ Linearization through binary variables
$\Longrightarrow$ MILP

A MILP to solve the discretized problem

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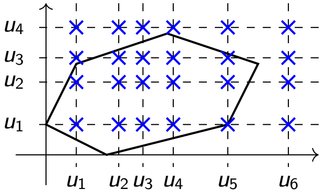

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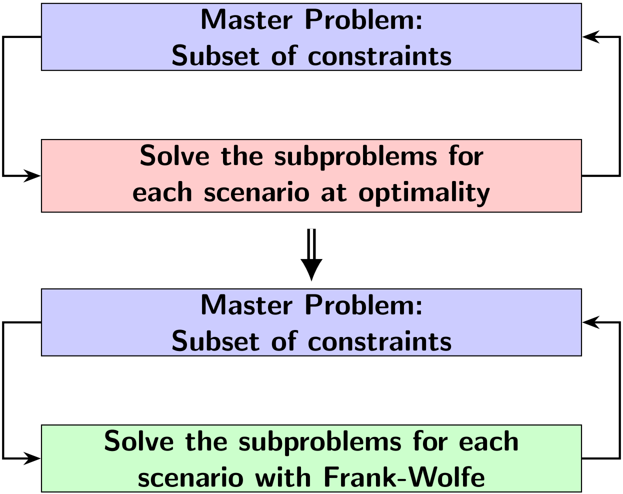
Adapted Benders' Algorithm



Contents

KKT reformulation

$$\begin{aligned} \max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \quad & d^\top y + c^\top x + \|x\|_2^2 \\ \text{s.t.} \quad & Ax + Ty = b, \\ & x \geq 0,\ y \geq 0 \end{aligned}$$

KKT conditions $\implies$ MILP.

→ **Advantage:** Provide optimal solution.

→ **Drawback:** Large number of binary variables.

→ **Worst than Gurobi solving directly the quadratic problem**

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PSD relaxation

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PSD relaxation first order:

$$
\begin{aligned}
\max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m_+} \quad & d^\top y + c^\top x + Tr(X) \\
\text{s.t.} \quad & Ax + Ty = b, \\
& y \geq 0,\ X = \begin{bmatrix} 1 & x^\top \\ x & xx^\top \end{bmatrix}
\end{aligned}
$$

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$$\begin{aligned} \max_{y\in\mathbb{R}^m_+} \quad & d^\top y + c^\top X_{1,.} + Tr(X) \\ \text{s.t.} \quad & AX_{1,.} + Ty = b \\ & X \succeq 0 \\ & y \geq 0 \end{aligned}$$

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- $\rightarrow$ **Advantage:** Tractable convex problem.
- $\rightarrow$ **Drawback:** Optimal value: $+\infty$
- $\rightarrow$ **Worst than Gurobi solving directly the quadratic problem**

PSD relaxation

$$\max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m_+} \quad d^\top y + c^\top x + \|x\|_2^2$$
$$\text{s.t.} \quad Ax + Ty = b,$$
$$x \geq 0,\ y \geq 0$$

PSD relaxation second order:

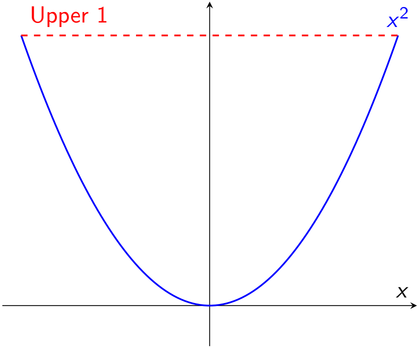
Idea: Consider a large PSD matrix that include product variables between x and y up to order 2.

→ **Advantage:** Bounded convex problem and good relaxation.

→ **Drawback:** Untractable with solvers like Mosek.

→ **Worst than Gurobi solving directly the quadratic problem**

Piecewise linear upper approximation



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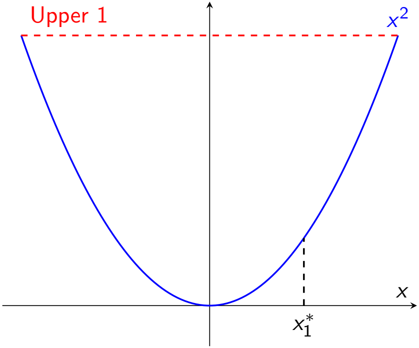

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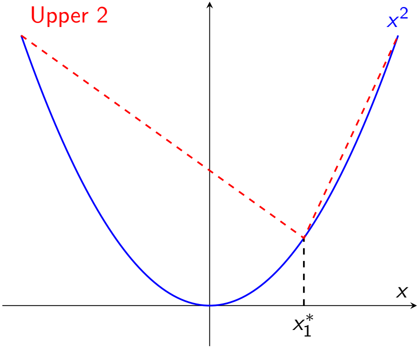

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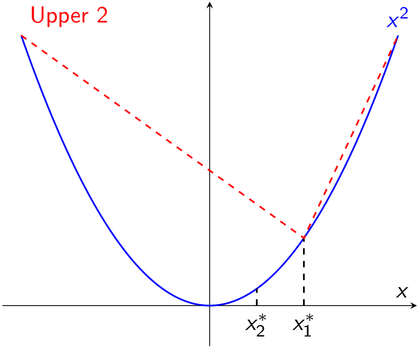

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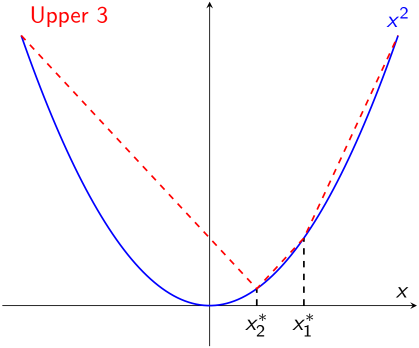

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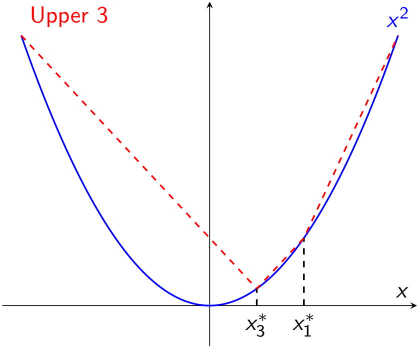

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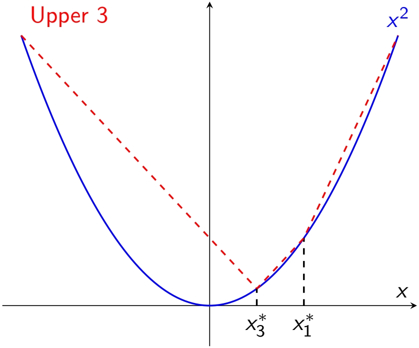
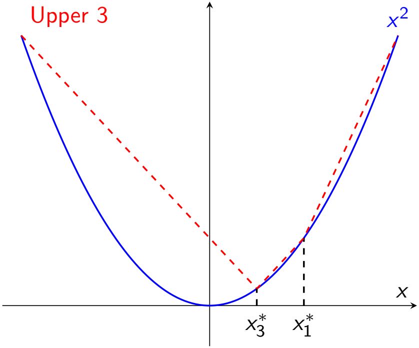


MILPs.

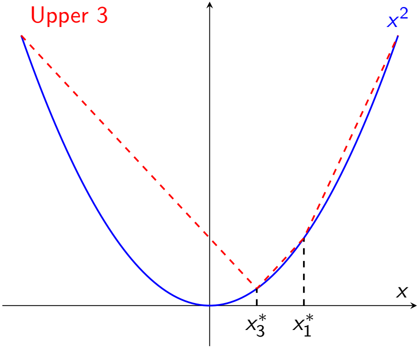
Piecewise linear upper approximation



MILPs.

Number of binary variables increases at each iteration

Piecewise linear upper approximation



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Convergence in a few iterations.

Contents

Instances

Source: SMS++/ EDF. Instances with generators (10, 50, 100) and only one random demand (dimension: $T = 24$)

Source IEEE: instance with network constraints:

54 generators and 10 wind farms $\Longrightarrow$ dimension of uncertainty: $10 \times T = 240$

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Computational times

Units	10	50	100	54
ξ dimension	24	24	24	240
FW $(x_0 = 0)$	0.002s	0.002s	0.002s	0.04s
FW MILP $(x_0 \in \{0, M\}^T)$	0.01s	0.05s	0.1s	0.2s
MILP UB	0.1s	0.3s	0.5s	4s
Gurobi	0.2s	0.5s	0.75s	5s

Table: Time comparison (s)

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Number of binary variables: O(dimension of ξ $\times$ Number of pieces)

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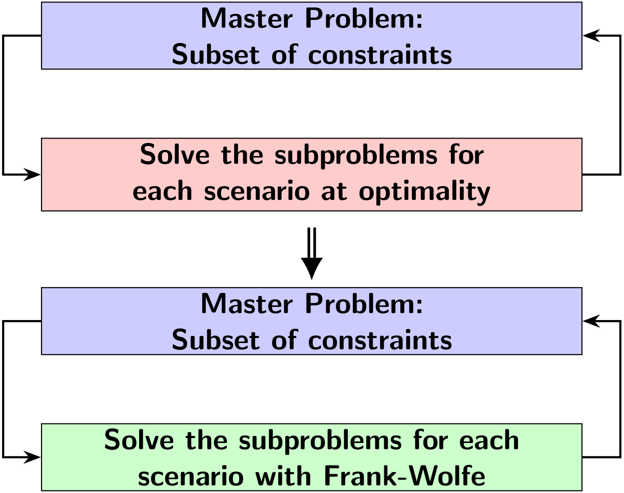
Convergence Analysis



Convergence Analysis

Units	10	50	100	54
ξ dimension	24	24	24	240
FW $(x_0 = 0)$	**3s**	**15s**	**40s**	**60s**
FW MILP $(x_0 \in \{0, M\}^T)$	30s	167s	400s	1400s
Gurobi	35s	300s	410s	-

Table: Computational time with 25 scenarios (0.1% gap)

Units	10	50	100	54
ξ dimension	24	24	24	240
FW $(x_0 = 0)$	0.3%	0.2%	0.3%	0.15%
FW MILP $(x_0 \in \{0, M\}^T)$	$< 0.1\%$	$< 0.1\%$	$< 0.1\%$	0.2%
Gurobi	$< 0.1\%$	$< 0.1\%$	$< 0.1\%$	-

Table: Real gap after "convergence" (25 scenarios)

Conclusion

Summary

- Analysis of an NP-hard problem that arises as the oracle problem in a Benders' decomposition for solving a DRO problem.
- Show how the Frank-Wolfe algorithm can be used to progress in the Benders' algorithm.
- Exploration of methods for calculating upper bounds.
- Evaluate the performance of the Adapted Benders' decomposition.